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# A class of nilpotent evolution algebras

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## ABSTRACT

Recently, by A. Elduque and A. Labra a new technique and a type of an evolution algebra are introduced. Several nilpotent evolution algebras defined in terms of bilinear forms and symmetric endomorphisms are constructed. The technique then used for the classification of the nilpotent evolution algebras up to dimension five. In this article, we develop this technique for high dimensional evolution algebras. We construct nilpotent evolution algebras of any type. Moreover, we show that, except the cases considered by Elduque and Labra, this construction of nilpotent evolution algebras does not give all possible nilpotent evolution algebras.

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## 1. Introduction

An evolution algebra over a field is an algebra with a basis on which multiplication is defined by the product of distinct basis terms being zero and the square of each basis element being a linear form in basis elements [13].

In study of any class of algebras, it is important to describe up to isomorphism at least algebras of lower dimensions. In [10] and [14], the classifications of associative and nilpotent Lie algebras of low dimensions were given.

About classifications of evolution algebras and their nilpotency the following results are known:

In [3] (see also [5]) two-dimensional evolution algebras over the complex numbers were classified. For the classification of two-dimensional evolution algebras over the real numbers see [11]. Moreover, in [5] a class of algebras isomorphic to evolution algebras with Jordan form matrices are considered and a criterion for their nilpotency is given. In case when the eigenvalue of the corresponding Jordan block is zero, the article gives the criterion of nilpotency of the corresponding finite-dimensional complex evolution algebras. In [5] it is shown that for nilpotent  $n$ -dimensional complex evolution algebras the possible maximal nilpotency index is  $2^{n-1} + 1$ . The article [4] gives classification of finite-dimensional complex evolution algebras with maximal nilpotent index  $2^{n-1} + 1$ .

Recently, in [2] the authors classified three-dimensional evolution algebras over a field having characteristic different from 2 and in which there are roots of orders 2, 3, and 7. It is proved that there are 116 types of three-dimensional evolution algebras.

Very recently, in [8] the authors studied the distribution of finite-dimensional evolution algebras over any base field into isotopism<sup>1</sup> classes according to their structure tuples and to the dimension of their annihilators. It is shown the existence of four isotopism classes of two-dimensional evolution algebras, whatever the base field is. For the three-dimensional case, it is shown how to deal with the distribution into isotopism classes of evolution algebras of higher dimensions.

In [6], it is shown that a finite-dimensional evolution algebra is nilpotent if and only if the associated graph contains no oriented cycles. This result is equivalent to the fact that, for some reordering of the given natural basis, the corresponding matrix of structural coefficients are upper triangular [3]. Thus, the nilpotency of a finite-dimensional evolution algebra can be seen by the graph associated to the natural basis.

In [7] (see also [9]), a classification of indecomposable nilpotent evolution algebras up to dimension five over algebraically closed fields of characteristic not two is given. To do this in [7] the type and several invariant subspaces related to the upper annihilating series of finite-dimensional nilpotent evolution algebras are introduced. A class of nilpotent evolution algebras, defined in terms of a nondegenerate, symmetric, bilinear form and some commuting, symmetric, diagonalizable endomorphisms relative to the form, are constructed.

In [12], a nilpotency condition on the evolution algebra that corresponds to a permutation is given.

In this article, we develop the methods of [7] for high dimensional evolution algebras. We construct nilpotent evolution algebras of any type. We show that, except the cases considered by Elduque and Labra, this construction of nilpotent evolution algebras does not give all possible nilpotent evolution algebras.

## 2. Basic definitions and facts

*Evolution algebras.* Let  $(E, \cdot)$  be an algebra over a field  $K$ . If it admits a basis  $\{e_1, e_2, \dots\}$ , such that

$$e_i \cdot e_j = \begin{cases} 0, & \text{if } i \neq j; \\ \sum_k a_{ik} e_k, & \text{if } i = j, \end{cases}$$

then this algebra is called an *evolution algebra* [13]. The basis is called a natural basis. We denote by  $A = (a_{ij})$  the matrix of the structural constants of the evolution algebra  $E$ .

It is known that an evolution algebra is commutative and not associative, in general. For basic properties of the evolution algebra see [13].

For an evolution algebra  $E$  and  $k \geq 1$  we introduce the following sequence

$$E^k = \sum_{i=1}^{k-1} E^i E^{k-i}. \quad (2.1)$$

Since  $E$  is a commutative algebra we obtain

$$E^k = \sum_{i=1}^{\lfloor k/2 \rfloor} E^i E^{k-i},$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ .

**Definition 1.** An evolution algebra  $E$  is called *nilpotent* if there exists some  $n \in \mathbb{N}$  such that  $E^n = 0$ . The smallest  $n$  such that  $E^n = 0$  is called the *index of nilpotency*.

<sup>1</sup>The concept of isotopism of algebras was introduced in [1] as a generalization of isomorphism. Two  $n$ -dimensional algebras  $A$  and  $B$  defined over a field  $K$  are isotopic if there exist three non-singular linear transformations  $f, g$  and  $h$  from  $A$  to  $B$  such that  $f(u)g(v) = h(uv)$ , for all  $u, v \in A$ .

The following theorem is known (see [3]).

**Theorem 1.** *An  $n$ -dimensional evolution algebra  $E$  is nilpotent iff the matrix of the structural constants corresponding to  $E$  can be written as*

$$\hat{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.2}$$

*Upper annihilating series.* Following [7] we introduce the following definitions:

Let  $\mathcal{E}$  be an evolution algebra with a natural basis  $B = \{e_1, \dots, e_n\}$  and matrix of structural constants  $A = (a_{ij})$ . The graph  $\Gamma(\mathcal{E}, B) = (V, E)$ , with  $V = \{1, \dots, n\}$  and  $E = \{(i, j) \in V \times V : a_{ij} \neq 0\}$ , is called the graph attached to the evolution algebra  $\mathcal{E}$  relative to the natural basis  $B$ .

**Definition 2.** *For an algebra  $\mathcal{A}$  define the chain  $\text{ann}^i(\mathcal{A}), i \geq 1$  by*

$$\begin{aligned} \text{ann}^1(\mathcal{A}) &:= \text{ann}(\mathcal{A}) := \{x \in \mathcal{A} : x\mathcal{A} = \mathcal{A}x = 0\}, \\ \text{ann}^i(\mathcal{A})/\text{ann}^{i-1}(\mathcal{A}) &:= \text{ann}(\mathcal{A}/\text{ann}^{i-1}(\mathcal{A})). \end{aligned}$$

**Definition 3.** *The following series is called the upper annihilating series:*

$$0 = \text{ann}^0(\mathcal{A}) \subseteq \text{ann}^1(\mathcal{A}) \subseteq \dots \subseteq \text{ann}^r(\mathcal{A}) \subseteq \dots$$

It is known that a non-associative algebra (in particular an evolution algebra) is nilpotent if and only if its upper annihilating series reaches  $\mathcal{A}$ , i.e.,  $\text{ann}^r(\mathcal{A}) = \mathcal{A}$ , for some  $r \geq 1$  [7].

**Definition 4.** *Let  $\mathcal{A}$  be a finite-dimensional nilpotent non-associative algebra over a field  $\mathbb{F}$ , and let  $r$  be the lowest natural number such that  $\text{ann}^r(\mathcal{A}) = \mathcal{A}$ . The type of  $\mathcal{A}$  is the sequence  $[n_1, n_2, \dots, n_r]$  such that  $n_1 + n_2 + \dots + n_i = \dim_{\mathbb{F}}(\text{ann}^i(\mathcal{A}))$ , for all  $i = 1, 2, \dots, r$ . Thus*

$$n_i = \dim_{\mathbb{F}}(\text{ann}^i(\mathcal{A})) - \dim_{\mathbb{F}}(\text{ann}^{i-1}(\mathcal{A})), i = 1, 2, \dots, r.$$

### 3. Nilpotent evolution algebras

Consider a field  $\mathbb{F}$  of characteristic not equal to 2. Let  $\mathcal{U}$  be a vector space over  $\mathbb{F}$  with  $\dim_{\mathbb{F}}\mathcal{U} = n$ .

**Definition 5.** *Let  $b : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{F}$  be a nondegenerate symmetric bilinear form and let  $f_i : \mathcal{U} \rightarrow \mathcal{U}, i = 1, 2, \dots, k-1$  be pairwise commuting, symmetric (relative to  $b$ ), diagonalizable endomorphisms. We define the algebra  $E(\mathcal{U}, b, f_1, \dots, f_{k-1}) := \mathcal{U} \times \underbrace{\mathbb{F} \times \dots \times \mathbb{F}}_k$  with multiplication*

$$(u, \alpha_1, \dots, \alpha_k)(v, \beta_1, \dots, \beta_k) = (0, b(u, v), b(f_1(u), v) + \alpha_1\beta_1, b(f_2(u), v) + \alpha_2\beta_2, \dots, b(f_{k-1}(u), v) + \alpha_{k-1}\beta_{k-1}), \tag{3.1}$$

for any  $u, v \in \mathcal{U}$  and  $\alpha_i, \beta_j \in \mathbb{F}$ .

**Proposition 1.**  *$E(\mathcal{U}, b, f_1, \dots, f_{k-1})$  is a nilpotent evolution algebra of type  $\underbrace{[1, 1, \dots, 1]}_k, n$ .*

*Proof.* By assumptions there is an orthogonal basis  $\{u_1, \dots, u_n\}$  of  $\mathcal{U}$ , relative to  $b$ , consisting of common eigenvalues for  $f_i, i = 1, \dots, k-1$ . Then

$$e_1 = (u_1, 0, 0, \dots, 0), \dots, e_n = (u_n, 0, 0, \dots, 0), \\ e_{n+1} = (0, 1, 0, 0, \dots, 0), \dots, e_{n+k} = (0, 0, \dots, 0, 1)$$

is a natural basis of  $E = E(\mathcal{U}, b, f_1, \dots, f_{k-1})$ . Moreover,  $e_i e_j = 0$  if  $i \neq j$  and

$$e_i^2 = \begin{cases} \sum_{j=1}^k \lambda_{ij} e_{n+j}, & \text{if } i = 1, \dots, n \\ e_{i+1}, & \text{if } i = n+1, \dots, n+k-1 \\ 0, & \text{if } i = n+k, \end{cases} \quad (3.2)$$

where

$$\lambda_{ij} = \begin{cases} b(u_i, u_i), & \text{if } j = 1 \\ b(f_{j-1}(u_i), u_i), & \text{if } j = 2, \dots, k. \end{cases}$$

Now using multiplication (3.1) we calculate  $\text{ann}^j(E)$ . We have

$$\text{ann}^j(E) = \{(u, \alpha_1, \dots, \alpha_k) \in E : (u, \alpha_1, \dots, \alpha_k)(v, \beta_1, \dots, \beta_k) = 0, \forall (v, \beta_1, \dots, \beta_k) \in E\} \\ = \underbrace{0 \times 0 \times \dots \times 0}_k \times \mathbb{F}.$$

Note that the first zero in the RHS of this formula is  $n$ -dimensional zero-vector.

$$\text{ann}^2(E) = \{(u, \alpha_1, \dots, \alpha_k) \in E : \\ (u, \alpha_1, \dots, \alpha_k)(v, \beta_1, \dots, \beta_k) \in \text{ann}^j(E), \forall (v, \beta_1, \dots, \beta_k) \in E\} \\ = \underbrace{0 \times 0 \times \dots \times 0}_{k-1} \times \mathbb{F} \times \mathbb{F}.$$

Using mathematical induction over  $j$  one can prove that

$$\text{ann}^j(E) = \underbrace{0 \times 0 \times \dots \times 0}_{k-j+1} \times \underbrace{\mathbb{F} \times \dots \times \mathbb{F}}_j, j = 1, \dots, k.$$

and  $\text{ann}^{k+1}(E) = E$ . Thus we have

$$n_i = \dim_{\mathbb{F}}(\text{ann}^i(E)) - \dim_{\mathbb{F}}(\text{ann}^{i-1}(E)) = \begin{cases} 1, & \text{if } i = 1, \dots, k \\ n, & \text{if } i = k+1. \end{cases}$$

The following proposition shows that Proposition 1 does not give all nilpotent evolution algebras of type  $\underbrace{[1, \dots, 1, n]}_k$ .

**Proposition 2.** For each  $k \geq 4$  and  $n \geq 1$  there is an evolution algebra  $E'$  of type  $\underbrace{[1, \dots, 1, n]}_k$ , which is non-isomorphic to  $E = E(\mathcal{U}, b, f_1, \dots, f_{k-1})$  for any collection  $(\mathcal{U}, b, f_1, \dots, f_{k-1})$  as in Definition 5.

*Proof.* Let  $E'$  be of type  $\underbrace{[1, \dots, 1, n]}_k$  and  $\{h_1, \dots, h_n, h_{n+1}, \dots, h_{n+k}\}$  be a natural basis of this algebra. Moreover,

$$\text{ann}^j(E') = \underbrace{0 \times 0 \times \dots \times 0}_{k-j+1} \times \underbrace{\mathbb{F} \times \dots \times \mathbb{F}}_j, j = 1, \dots, k.$$

and  $\text{ann}^{k+1}(E') = E'$ . Then we choose  $E'$  such that  $h_i h_j = 0$  if  $i \neq j$  and

$$h_i^2 = \begin{cases} h_{n+1}, & \text{if } i = 1, \dots, n \\ h_{i+1} + h_{i+2}, & \text{if } i = n + 1, n + 2, \dots, n + k - 2 \\ h_{n+k}, & \text{if } i = n + k - 1 \\ 0, & \text{if } i = n + k. \end{cases} \tag{3.3}$$

We shall show that there is no a change from basis  $\{h_i\}$  (of algebra  $E'$ ) with multiplication (3.3) to the basis  $\{e_i\}$  (of algebra denoted by  $E$ ) with multiplication (3.2). We note that if such  $\varphi$ , (where  $\det\varphi \neq 0$ ) exists then

$$\varphi(\text{ann}^m(E')) = \text{ann}^m(E), 1 \leq m \leq k + 1.$$

Moreover, by Corollary 3.6 of [7],  $\varphi$  has the following block structure:

$$\begin{pmatrix} * & 0 & 0 & \dots & * \\ 0 & * & 0 & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & *, \end{pmatrix}$$

i.e.

$$\varphi(h_i) = \gamma_i e_i + \delta_i e_{n+k}, \gamma_i, \delta_i \in \mathbb{F}.$$

Thus  $\det(\varphi) = \gamma_1 \dots \gamma_{n+k}$ . From (3.3) we get

$$h_i^2 h_{i+1}^2 = h_{i+2}^2, i = n + 1, n + 2, \dots, n + k - 2.$$

Consequently,

$$\begin{aligned} \varphi(h_{i+2}^2) &= \varphi(h_i^2) \varphi(h_{i+1}^2) = \varphi(h_i)^2 \varphi(h_{i+1})^2 = (\gamma_i e_i + \delta_i e_{n+k})^2 (\gamma_{i+1} e_{i+1} + \delta_{i+1} e_{n+k})^2 \\ &= \gamma_i^2 e_i^2 \gamma_{i+1}^2 e_{i+1}^2 = \gamma_i^2 \gamma_{i+1}^2 e_{i+1} e_{i+2} = 0 \end{aligned}$$

for each  $i = n + 1, n + 2, \dots, n + k - 2$ . Using this we obtain

$$0 = \varphi(h_{i+2}^2) = \varphi(h_{i+2})^2 = \gamma_{i+2}^2 e_{i+2}^2 = \gamma_{i+2}^2 e_{i+3},$$

hence  $\gamma_{i+2} = 0$  for each  $i = n + 1, \dots, n + k - 3$ . Thus if  $k \geq 4$  then  $\det(\varphi) = 0$ , i.e. there is no isomorphism between  $E'$  and  $E$ . □

**Remark 1.** In [7] for  $k \leq 3$  it is shown that any algebra of type  $\underbrace{[1, \dots, 1, n]}_k$  is isomorphic to

$E(\mathcal{U}, b, f_1, \dots, f_{k-1})$  for some  $(\mathcal{U}, b, f_1, \dots, f_{k-1})$ . Proposition 2 shows that this kind of result is not true for any  $k > 3$ .

**Definition 6.** Let  $b : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{F}$  be a nondegenerate, symmetric, bilinear form and let  $0 \neq u \in \mathcal{U}$ . Define the algebra

$$E_{lr}(\mathcal{U}, b, u) := \underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_l \times \mathcal{U} \times \underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_r$$

with multiplication

$$\begin{aligned} &(\alpha_1, \dots, \alpha_l, x, \beta_1, \dots, \beta_r)(\gamma_1, \dots, \gamma_l, y, \delta_1, \dots, \delta_r) = \\ &(0, \alpha_1 \gamma_1, \alpha_2 \gamma_2, \dots, \alpha_{l-1} \gamma_{l-1}, \alpha_l \gamma_l u, b(x, y), \beta_1 \delta_1, \dots, \beta_{r-1} \delta_{r-1}), \end{aligned} \tag{3.4}$$

for any  $x, y \in \mathcal{U}$  and  $\alpha_i, \beta_j, \gamma_k, \delta_m \in \mathbb{F}$ .

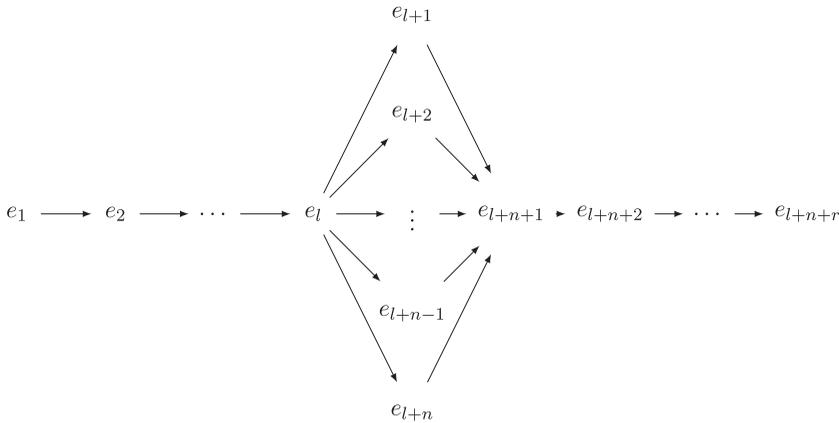


Figure 1. The graph of the evolution algebra given by multiplication (3.5).

**Theorem 2.**  $E_{lr}(\mathcal{U}, b, u)$  is a nilpotent evolution algebra of type  $[1, 1, \dots, 1, n, 1, 1, \dots, 1]$ .

*Proof.* Let  $\{u_1, \dots, u_n\}$  be an orthogonal basis of  $\mathcal{U}$  related to  $b$ . Then using (3.4) it is easy to see that

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_l = (\underbrace{0, \dots, 0}_l, 1, 0, \dots, 0), \\ e_{l+1} &= (\underbrace{0, \dots, 0}_l, u_1, \underbrace{0, \dots, 0}_r), \dots, e_{l+n} = (\underbrace{0, \dots, 0}_l, u_n, \underbrace{0, \dots, 0}_r), \\ e_{l+n+1} &= (0, \dots, 0, \underbrace{1, 0, \dots, 0}_r), \dots, e_{l+n+r} = (0, \dots, 0, \underbrace{0, 0, \dots, 1}_r) \end{aligned}$$

is a natural basis of  $E = E_{lr}(\mathcal{U}, b, u)$  making it an evolution algebra. Moreover,  $e_i e_j = 0$  if  $i \neq j$ . If  $u = \sum_{k=1}^n c_k u_k$  and  $\lambda_k = b(u_k, u_k)$  then

$$e_i^2 = \begin{cases} e_{i+1}, & \text{if } i = 1, \dots, l-1 \text{ and } i = l+n+1, l+n+2, \dots, l+n+r-1 \\ \sum_{k=1}^n c_k e_{l+k}, & \text{if } i = l \\ \lambda_j e_{l+n+1}, & \text{if } i = l+j, j = 1, \dots, n \\ 0, & \text{if } i = l+n+r. \end{cases} \tag{3.5}$$

In Figure 1, the graph of the evolution algebra (see page 3 for definition) with multiplication (3.5) is given.

Now we check the type of this evolution algebra. From (3.4) it follows that

$$\text{ann}(E) = \underbrace{0 \times 0 \times \dots \times 0}_{l+r} \times \mathbb{F};$$

Note that in the last formula and below one 0 is  $n$ -dimensional zero-vector.

$$\begin{aligned} \text{ann}^2(E) &= \{(\alpha_1, \dots, \alpha_l, x, \beta_1, \dots, \beta_r) \in E : \\ &(\alpha_1, \dots, \alpha_l, x, \beta_1, \dots, \beta_r)(\gamma_1, \dots, \gamma_l, y, \delta_1, \dots, \delta_r) = \\ &(0, \alpha_1 \gamma_1, \alpha_2 \gamma_2, \dots, \alpha_{l-1} \gamma_{l-1}, \alpha_l \gamma_l u, b(x, y), \beta_1 \delta_1, \dots, \beta_{r-1} \delta_{r-1}) \in \underbrace{0 \times 0 \times \dots \times 0}_{l+r} \times \mathbb{F}, \end{aligned}$$

$$\text{for any } (\gamma_1, \dots, \gamma_l, y, \delta_1, \dots, \delta_r) \in E\} = \underbrace{0 \times 0 \times \dots \times 0}_{l+r-1} \times \mathbb{F} \times \mathbb{F};$$

Similarly by mathematical induction over  $j$  one shows that

$$\begin{aligned} \text{ann}^j(E) &= \underbrace{0 \times 0 \times \cdots \times 0}_{l+r-j+1} \times \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_j, \quad j = 1, \dots, r. \\ \text{ann}^{r+1}(E) &= \{(\alpha_1, \dots, \alpha_l, x, \beta_1, \dots, \beta_r) \in E : \\ &(\alpha_1, \dots, \alpha_l, x, \beta_1, \dots, \beta_r)(\gamma_1, \dots, \gamma_l, y, \delta_1, \dots, \delta_r) \\ &= (0, \alpha_1\gamma_1, \alpha_2\gamma_2, \dots, \alpha_{l-1}\gamma_{l-1}, \alpha_l\gamma_l u, b(x, y), \beta_1\delta_1, \dots, \beta_{r-1}\delta_{r-1}) \in \\ &\underbrace{0 \times 0 \times \cdots \times 0}_{l+1} \times \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_r, \text{ for any } (\gamma_1, \dots, \gamma_l, y, \delta_1, \dots, \delta_r) \in E\} \\ &= \underbrace{0 \times 0 \times \cdots \times 0}_l \times \mathcal{U} \times \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_r; \end{aligned}$$

Now using induction over  $m$  one can show that

$$\text{ann}^{r+1+m}(E) = \underbrace{0 \times 0 \times \cdots \times 0}_{l-m} \times \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_m \times \mathcal{U} \times \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_r, \quad m = 1, \dots, l.$$

Thus we have

$$n_i = \dim_{\mathbb{F}}(\text{ann}^i(E)) - \dim_{\mathbb{F}}(\text{ann}^{i-1}(E)) = \begin{cases} 1, & \text{if } i \neq r+1 \\ n, & \text{if } i = r+1. \end{cases} \quad \square$$

**Theorem 3.** *If  $l \geq 2$  or  $r \geq 2$  then for each  $n \geq 1$  there is an evolution algebra  $E$  of type  $\underbrace{[1, 1, \dots, 1]}_l, \underbrace{[n, 1, 1, \dots, 1]}_r$  which is not isomorphic to an algebra  $E_r(\mathcal{U}, b, u)$ , for any  $(\mathcal{U}, b, u)$  as in*

*Definition 6.*

*Proof.* Let  $B = \{h_1, \dots, h_{l+n+r}\}$  be a natural basis of  $E$  with type  $\underbrace{[1, 1, \dots, 1]}_l, \underbrace{[n, 1, 1, \dots, 1]}_r$ . Then

$$\text{ann}^j(E) = \begin{cases} \bigoplus_{i=1}^j \mathbb{F}e_{l+n+r-i+1}, & \text{if } j = 1, \dots, r \\ \bigoplus_{i=1}^n \mathbb{F}e_{l+i} \oplus_{s=1}^r \mathbb{F}e_{l+n+s}, & \text{if } j = r+1 \\ \bigoplus_{t=1}^m \mathbb{F}e_{l-t+1} \oplus_{i=1}^n \mathbb{F}e_{l+i} \oplus_{s=1}^r \mathbb{F}e_{l+n+s}, & \text{if } j = r+1+m, \quad m = 1, \dots, l. \end{cases}$$

Using these formulas we get the following multiplication table:

$$\begin{aligned} h_i^2 &= \sum_{j=i+1}^{l+n+r} \alpha_{i,j} h_j, \quad \text{if } i = 1, \dots, l \\ h_{l+t}^2 &= \sum_{j=l+n+1}^{l+n+r} \alpha_{l+t,j} h_j, \quad \text{if } t = 1, \dots, n \\ h_{l+n+m}^2 &= \sum_{j=l+n+m+1}^{l+n+r} \alpha_{l+n+m,j} h_j, \quad \text{if } m = 1, \dots, r-1 \\ h_{l+n+r}^2 &= 0. \end{aligned} \tag{3.6}$$

Thus the matrix of structural constants of this algebra has upper triangular form with zeros in the diagonal (see Theorem 1).

Case:  $r \geq 2$ . Here we consider a particular case of (3.6), i.e., we take

$$\begin{aligned} h_i^2 &= \sum_{j=i+1}^{l+n+r} \alpha_{i,j} h_j, \quad \text{if } i = 1, \dots, l \\ h_{l+1}^2 &= c h_{l+n+r}, \quad c \neq 0 \\ h_{l+t}^2 &= \sum_{j=l+n+1}^{l+n+r} \alpha_{l+t,j} h_j, \quad \text{if } t = 2, \dots, n \\ h_{l+n+m}^2 &= \sum_{j=l+n+m+1}^{l+n+r} \alpha_{l+n+m,j} h_j, \quad \text{if } m = 1, \dots, r-1 \\ h_{l+n+r}^2 &= 0. \end{aligned} \quad (3.7)$$

We assume that there exists a change  $\psi$  of basis  $\{h_1, \dots, h_{l+n+r}\}$ , of algebra  $E$ , with multiplication (3.7) to the basis  $\{e_1, \dots, e_{l+n+r}\}$ , of algebra  $E'$ , with multiplication (3.5). Since  $\psi(\text{ann}^j(E)) = \text{ann}^j(E')$ , we get for  $\psi$  the following equalities:

$$\psi(h_i) = \begin{cases} \sum_{j=i}^{l+n+r} \gamma_{ij} e_j, & i = 1, \dots, l; \\ \sum_{j=l+1}^{l+n+r} \gamma_{ij} e_j, & i = l+1, \dots, l+n; \\ \sum_{j=i}^{l+n+r} \gamma_{ij} e_j, & i = l+n+1, \dots, l+n+r. \end{cases}$$

By these formulas we have

$$\det \psi = \det(\gamma_{ij})_{i,j=1}^l \cdot \det(\gamma_{ij})_{i,j=l+1}^{l+n} \cdot \det(\gamma_{ij})_{i,j=l+n+1}^{l+n+r}.$$

For any  $p \in \{l+1, \dots, l+n\}$ , we have

$$\begin{aligned} \psi(h_p h_{l+n+r-1}) &= \psi(h_p) \psi(h_{l+n+r-1}) = \sum_{j=l+1}^{l+n+r} \gamma_{pj} \gamma_{l+n+r-1,j} e_j^2 \\ &= \sum_{j=l+1}^{l+n+r-2} \gamma_{pj} \gamma_{l+n+r-1,j} e_j^2 + \gamma_{p,l+n+r-1} \gamma_{l+n+r-1,l+n+r-1} e_{l+n+r} = 0. \end{aligned}$$

In particular, from the last equality we get the following system of equations

$$\gamma_{p,l+n+r-1} \gamma_{l+n+r-1,l+n+r-1} = 0, \quad \text{for all } p \in \{l+1, \dots, l+n\}.$$

We note that  $\gamma_{l+n+r-1,l+n+r-1} \neq 0$ , consequently

$$\gamma_{p,l+n+r-1} = 0, \quad \text{for all } p \in \{l+1, \dots, l+n\}. \quad (3.8)$$

Now consider

$$\psi(h_{l+1}^2) = \psi(c h_{l+n+r}) = c \psi(h_{l+n+r}) = c \gamma_{l+n+r,l+n+r} e_{l+n+r}. \quad (3.9)$$

Since  $\det \psi \neq 0$  we have  $\gamma_{l+n+r,l+n+r} \neq 0$ . On the other hand we have

$$\psi(h_{l+1}^2) = \psi(h_{l+1})^2 = \left( \sum_{j=l+1}^{l+n+r} \gamma_{l+1,j} e_j \right)^2 = \sum_{j=l+1}^{l+n+r-2} \gamma_{l+1,j}^2 e_j^2 + \gamma_{l+1,l+n+r-1} e_{l+n+r}. \quad (3.10)$$

By (3.8)–(3.10) we get the following contradiction

$$0 \neq c\gamma_{l+n+r,l+n+r} = \gamma_{l+1,l+n+r-1} = 0.$$

Thus such  $\psi$  does not exist.

Case:  $l \geq 2$ . Here we consider the following particular case of (3.6):

$$\begin{aligned} h_i^2 &= h_{i+1} + h_{i+2}, \text{ if } i = 1, \dots, l \\ h_{l+t}^2 &= \sum_{j=l+n+1}^{l+n+r} \alpha_{l+t,j} h_j, \text{ if } t = 1, \dots, n \\ h_{l+n+m}^2 &= h_{l+n+m+1} + h_{l+n+m+2}, \text{ if } m = 1, \dots, r-2 \\ h_{l+n+r-1}^2 &= h_{l+n+r} \\ h_{l+n+r}^2 &= 0. \end{aligned} \tag{3.11}$$

Assume that there exists a change  $\varphi$  of basis  $\{e_1, \dots, e_{l+n+r}\}$ , of algebra  $E'$ , with multiplication (3.5) to the basis  $\{h_1, \dots, h_{l+n+r}\}$ , of algebra  $E$ , with multiplication (3.7). Then since

$$\varphi(\text{ann}^j(E')) = \text{ann}^j(E), j \geq 1,$$

we get

$$\varphi(e_i) = \sum_{j=i}^{l+n+r} \gamma_{ij} h_j, \quad i = 1, \dots, l.$$

The condition  $\det(\varphi) \neq 0$  implies  $\gamma_{11}\gamma_{22}\dots\gamma_{l+n+r,l+n+r} \neq 0$ .

From (3.5) we get

$$\begin{aligned} e_i^2 e_{i+1}^2 &= e_{i+1} e_{i+2} = 0, \quad i = 1, 2, \dots, l-2. \\ e_{i-1}^2 e_i^2 &= e_i \left( \sum_{k=1}^n c_k e_{l+k} \right) = 0. \\ e_i^2 e_{i+1}^2 &= \left( \sum_{k=1}^n c_k e_{l+k} \right) \lambda_1 e_{l+n+1} = 0. \\ e_{l+1}^2 e_{l+2}^2 &= \lambda_1 \lambda_2 e_{l+n+1}^2 = \lambda_1 \lambda_2 e_{l+n+2} \neq 0. \end{aligned}$$

Consequently,

$$\varphi(e_i^2 e_{i+1}^2) = 0, \text{ for each } i = 1, 2, \dots, l. \tag{3.12}$$

On the other hand we have

$$\begin{aligned} \varphi(e_i^2 e_{i+1}^2) &= \left( \sum_{j=i}^{l+n+r} \gamma_{ij} h_j \right)^2 \left( \sum_{j=i+1}^{l+n+r} \gamma_{i+1,j} h_j \right)^2 = \\ &= (\gamma_{ii}^2 h_i^2 + \dots) (\gamma_{i+1,i+1}^2 h_{i+1}^2 + \dots) = \\ &= (\gamma_{ii}^2 (h_{i+1} + h_{i+2}) + \dots) (\gamma_{i+1,i+1}^2 (h_{i+2} + h_{i+3}) + \dots) = \\ &= \gamma_{ii}^2 \gamma_{i+1,i+1}^2 h_{i+2}^2 + \dots, \text{ for each } i = 1, 2, \dots, l-1. \end{aligned} \tag{3.13}$$

By (3.12) from the last equality we get  $\gamma_{ii}\gamma_{i+1,i+1} = 0$  for each  $i = 1, \dots, l-1$ . Thus for  $l \geq 2$  we see that  $\det(\varphi) = 0$ . □

**Remark 2.** In [7] it was shown that if  $E$  is a nilpotent evolution algebra of type  $[1, n, 1]$  then  $E$  is isomorphic to an algebra  $E_{11}(\mathcal{U}, b, u)$ , (i.e.,  $l = r = 1$ ), for some  $(U, b, u)$  as in Definition 6. Theorem 3 shows that this result is not true for any  $l, r$  when at least one of them  $> 1$ .

**Definition 7.** Let  $b : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{F}$  be a non degenerate, symmetric, bilinear form. Define the algebra

$$E(\mathcal{U}, b) := \mathcal{U} \times \mathbb{F} \times \mathbb{F}$$

with multiplication

$$(u, \alpha_1, \alpha_2)(v, \beta_1, \beta_2) = (0, 0, b(u, v))$$

**Proposition 3.**  $E(\mathcal{U}, b)$  is a nilpotent evolution algebra of type  $[2, n]$ .

*Proof.* Let  $\{u_1, \dots, u_n\}$  be an orthogonal basis of  $\mathcal{U}$  related to  $b$ . Then  $\{e_i = (u_i, 0, 0), i = 1, \dots, n; e_{n+1} = (0, 1, 0), e_{n+2} = (0, 0, 1)\}$  is a natural basis of  $E = E(\mathcal{U}, b)$ . Moreover  $e_i e_j = 0, i \neq j$  and

$$e_i^2 = \lambda_i(0, 0, 1) = \lambda_i e_{n+2}, i = 1, \dots, n, \text{ with } \lambda_i = b(u_i, u_i), \\ e_{n+1}^2 = e_{n+2}^2 = 0.$$

It is easy to see that

$$\text{ann}(E) = 0 \times \mathbb{F} \times \mathbb{F}, \text{ann}^2(E) = E.$$

Thus  $n_1 = 2, n_2 = n$ . □

### 3.1. A construction of a nilpotent algebra of type $[n_1, n_2, \dots, n_k]$

Consider a field  $\mathbb{F}$  of characteristic not 2. Let  $\mathcal{U}_i$  be a vector space over  $\mathbb{F}$  with  $\dim_{\mathbb{F}} \mathcal{U}_i = m_i, i = 1, \dots, k$ . Denote by  $\{u_{i1}, \dots, u_{im_i}\}$  the basis elements of  $\mathcal{U}_i, i = 1, \dots, k$ .

**Definition 8.** Let  $\xi_i : \mathcal{U}_i \otimes \mathcal{U}_i \rightarrow \mathcal{U}_{i+1}, i = 1, \dots, k-1$  be symmetric bilinear mappings, such that  $\xi_i(u_{ip}, u_{iq}) = 0, p \neq q$ . We define the algebra

$$\mathbb{E} = \mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_k$$

with multiplication

$$(x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k) = \\ (0, \xi_1(x_1, y_1), \xi_2(x_2, y_2), \dots, \xi_{k-1}(x_{k-1}, y_{k-1})), \quad (3.14)$$

for any  $x_i, y_i \in \mathcal{U}_i$ .

**Theorem 4.** The algebra  $\mathbb{E} = \mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_k$  is a nilpotent evolution algebra of type  $[n_1, n_2, \dots, n_k]$ , with  $n_i = m_{k-i+1}$ .

*Proof.* We note that the following is a natural basis of  $\mathbb{E}$ :

$$\{e_{11}, e_{12}, \dots, e_{1m_1}; e_{21}, e_{22}, \dots, e_{2m_2}; \dots; e_{k1}, e_{k2}, \dots, e_{km_k}\},$$

where

$$e_{ij} = (0, \dots, 0, u_{ij}, 0, \dots, 0) \in \mathbb{E}, i = 1, \dots, k, j = 1, \dots, m_i.$$

Moreover, we have  $e_{ij} e_{pq} = 0, (i, j) \neq (p, q)$  and

$$e_{ij}^2 = (0, \dots, 0, \xi_i(u_{ij}, u_{ij}), 0, \dots, 0), i = 1, \dots, k-1, j = 1, \dots, m_i, \\ e_{kj}^2 = 0, j = 1, \dots, m_k$$

It is easy to see that

$$\begin{aligned} \text{ann}^1(\mathbb{E}) &= 0 \times 0 \times \cdots \times 0 \times \mathcal{U}_k, \text{ i.e., } \dim(\text{ann}^1(\mathbb{E})) = m_k. \\ \text{ann}^2(\mathbb{E}) &= 0 \times 0 \times \cdots \times 0 \times \mathcal{U}_{k-1} \times \mathcal{U}_k, \text{ i.e., } \dim(\text{ann}^2(\mathbb{E})) = \sum_{s=k-1}^k m_s. \\ &\dots\dots\dots \\ \text{ann}^j(\mathbb{E}) &= 0 \times 0 \times \cdots \times 0 \times \mathcal{U}_{k-j+1} \times \cdots \times \mathcal{U}_{k-1} \times \mathcal{U}_k, \text{ i.e., } \dim(\text{ann}^j(\mathbb{E})) = \sum_{s=k-j+1}^k m_s. \end{aligned}$$

Thus we have

$$n_i = \sum_{s=k-i+1}^k m_s - \sum_{s=k-i+2}^k m_s = m_{k-i+1}.$$

Then corresponding algebra is of type  $[n_1, n_2, \dots, n_k]$ . □

**Example 1.** Let  $\mathcal{V}_1$  be an evolution algebra with multiplication  $\xi_1 : \mathcal{V}_1 \otimes \mathcal{V}_1 \rightarrow \mathcal{V}_1$ , and  $\mathcal{V}_2 := \xi_1(\mathcal{V}_1, \mathcal{V}_1)$ . Assume a multiplication  $\xi_2$  is given on  $\mathcal{V}_2$  such that  $(\mathcal{V}_2, \xi_2)$  is an evolution algebra and  $\mathcal{V}_3 := \xi_2(\mathcal{V}_2, \mathcal{V}_2)$ . Consequently, define an evolution algebra  $(\mathcal{V}_{i-1}, \xi_{i-1})$  with  $\xi_{i-1}(\mathcal{V}_{i-1}, \mathcal{V}_{i-1}) = \mathcal{V}_i, i = 1, \dots, k$ . Consider

$$\mathcal{E} = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_k$$

with multiplication

$$(x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k) = (0, \xi_1(x_1, y_1), \xi_2(x_2, y_2), \dots, \xi_{k-1}(x_{k-1}, y_{k-1})), \tag{3.15}$$

for any  $x_i, y_i \in \mathcal{V}_i$ . Then by Theorem 4 the algebra  $\mathcal{E}$  is a nilpotent evolution algebra of type  $[n_1, n_2, \dots, n_k]$ , with  $n_i = \dim(\mathcal{V}_{k-i+1})$ .

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